UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

# $\mathcal{L}_1$-adaptive controller for a tractor-trailer trajectory following system

ME 562

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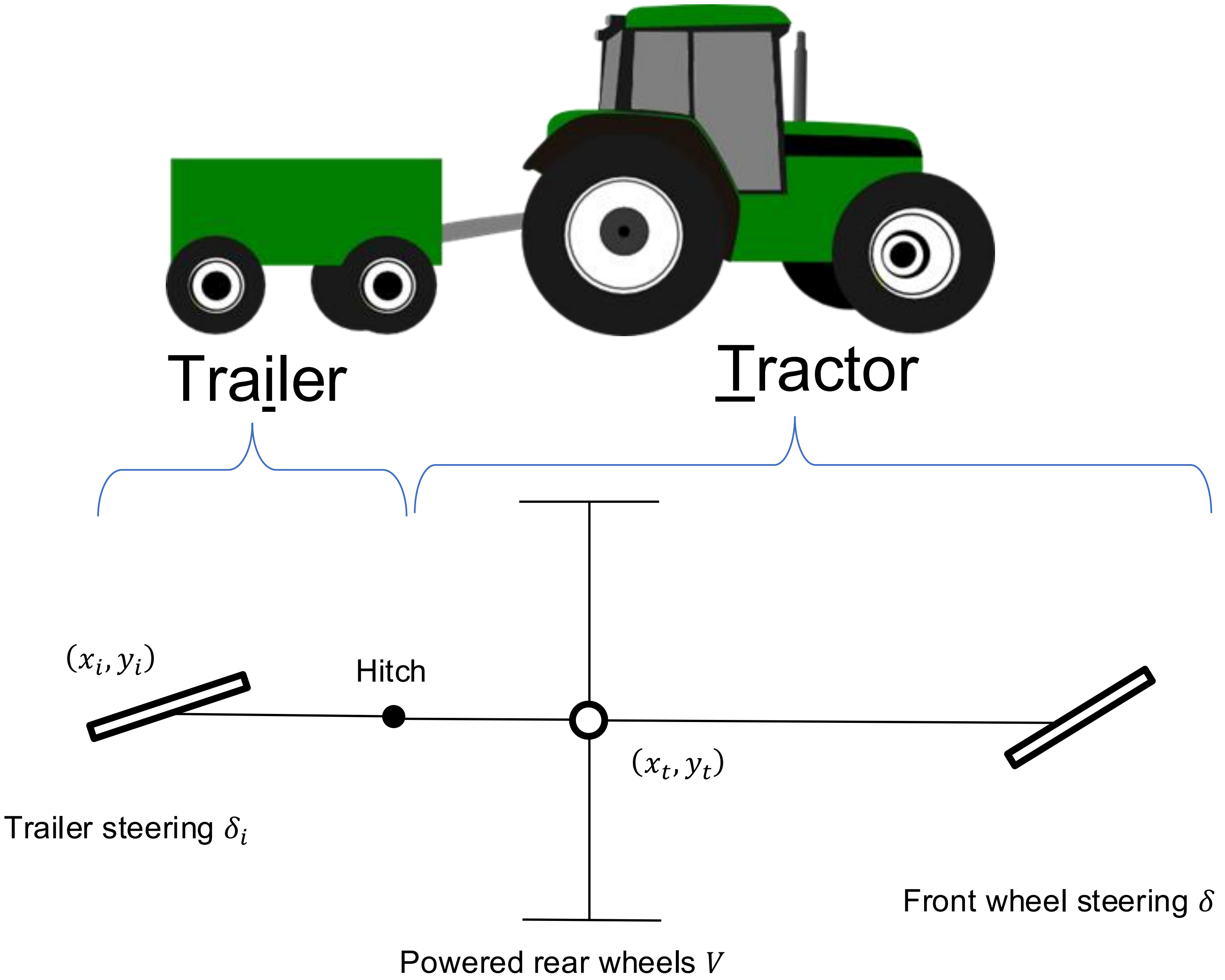
# Tractor-Trailer System



# Offset model and reference path tracking

## Stationary reference frame

$$\begin{cases} \dot{x}_t = (V - V_{lr})\cos(\theta_t) - V_{sr}\sin(\theta_t) \\ \dot{y}_t = (V - V_{lr})\sin(\theta_t) + V_{sr}\cos(\theta_t) \\ \dot{\theta}_t = \frac{V - V_{lr}}{l_1}\tan(\delta + \beta_f) + \frac{V_{sr}}{l_1} = \Omega_t \\ \dot{x}_i = \dot{x}_t + \Omega_t a\sin(\theta_t) + \Omega_i l_2 * \sin(\theta_i) \\ \dot{y}_i = \dot{x}_t + \Omega_t a\cos(\theta_t) + \Omega_i l_2 * \cos(\theta_i) \\ \dot{\phi} = \frac{1}{l_2\cos(\delta_i)}(-(V - V_{lr})\sin(\delta_i + \phi) + V_{sr}\cos(\delta_i + \phi) - \Omega_t(a\cos(\delta_i + \phi) + l_2\cos(\delta_i)) - V_{si}) = \theta_i - \theta_t \end{cases}$$

assume $\zeta = 1$, $V > V_{lr} > 0$

Require $R_1^2 > l_2^2 - a^2$

## Decoupled Error Model (rotating reference frame)

Tractor: $$\begin{cases} \dot{l_{os}} = -\sigma(V - V_{lr})\sin(\theta_{os}) - \sigma V_{sr}\cos(\theta_{os}) \\ \dot{\theta_{os}} = \frac{V - V_{lr}}{l_1}\tan(\delta + \beta_f) + \frac{V_{sr}}{l_1} - \sigma(V - V_{lr})\frac{\cos(\theta_{os})}{R_1 + l_{os}} + \sigma V_{sr}\frac{\sin(\theta_{os})}{R_1 + l_{os}} \end{cases}$$

Trailer: $$\{l_2\cos(\delta_i)\dot{\phi_{os}} = -(V - V_{lr})\sin(\delta_i + \phi_{os} + \phi_d) + V_{sr}\cos(\delta_i + \phi_{os} + \phi_d) - \left(\frac{V - V_{lr}}{l_1}\tan(\delta + \beta_f) + \frac{V_{sr}}{l_1}\right)(a\cos(\delta_i + \phi_{os} + \phi_d) + l_2\cos(\delta_i)) - V_{si}$$

# Indirect MRAC for the tractor with adaptive ISS backstepping

**Reference Model**:

$$\dot{x}_m = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} x_m + \begin{pmatrix} 0 \\ \omega_n^2 \end{pmatrix} r(t)$$

**Error terms**:

$$e := x_m - x \rightarrow \begin{matrix} e_1 := x_{m_1} - x_1 \\ e_2 := x_{m_2} - \xi_2 \end{matrix}$$

**Error Model**:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} e_2 - \Delta_1 \\ -\omega_n^2 x_{m_1} - 2\zeta\omega_n x_{m_2} + \omega_n^2 r(t) - f - \frac{1}{l_1} g u - \Delta_2 \end{pmatrix}$$

**1st order system**: $\dot{e}_1 = k_0 - \Delta_1$

$$V_0 = \frac{e_1^2}{2}, k_0 = -\mu e_1 = -(c_1 + c_2)e_1, \qquad \mu, c_1, c_2 > 0$$

$$\dot{V}_0 = -c_1 e_1^2 - c_2 e_1^2 - \Delta_1 e_1 - \frac{\Delta_1^2}{4c_2} + \frac{\Delta_1^2}{4c_2} = -c_1 e_1^2 - c_2\left(e_1 + \frac{\Delta_1}{2c_2}\right)^2 + \frac{\Delta_1^2}{4c_2}$$

$$\leq -c_1 e_1^2 + \frac{\Delta_1^2}{4c_2} = -\alpha_0(|e_1|) + \chi_0(|\Delta_1|)$$

Where $\Delta_1 \leq |V_{lr}| + |V_{sr}|$

**2nd order system**:

$$\dot{e}_1 = e_2 - \Delta_1$$

$$\dot{e}_2 = -\omega_n^2 x_{m_1} - 2\zeta\omega_n x_{m_2} + \omega_n^2 r(t) - f - \frac{1}{l_1} g k_1 - \Delta_2$$

$$V_1 = V_0 + \frac{1}{2}(e_2 - k_0)^2 + \frac{l_1}{2}\left(1 - \frac{\hat{l}_1}{l_1}\right)^2$$

Let $k_1 = u_1 + u_2$ and $\dot{\hat{l}}_1 = \tau_1 + \tau_2$

Use $(u_1, \tau_1)$ to deal with the undisturbed system, and $(u_2, \tau_2)$ to deal with $\Delta_1, \Delta_2$

Let $z = (e_2 - k_0), \beta = \left(e_1 - \omega_n^2 x_{m_1} - 2\zeta\omega_n x_{m_2} + \omega_n^2 r(t) - f + \mu e_2\right)$

$$u_1 = \frac{\hat{l}_1}{g}\left(e_1 - \omega_n^2 x_{m_1} - 2\zeta\omega_n x_{m_2} + \omega_n^2 r(t) - f + \mu e_2 + z\right)$$

$$u_2 = \frac{\hat{l}_1}{g}(z + z\mu^2)$$

$$\tau_1 = z\beta$$

$$\tau_2 = z^2(1 + \mu^2)$$

$$\dot{V} < -\mu e_1^2 - \frac{\hat{l}_1}{l_1} z^2 + \frac{\Delta_2^2}{4} + \frac{\Delta_1^2}{4} + \frac{\Delta_1^2}{4c_2} = -\alpha_1\left(\left|\begin{matrix} e_1 \\ e_2 \end{matrix}\right|\right) + \chi_1\left(\left|\begin{matrix} \Delta_1 \\ \Delta_2 \end{matrix}\right|\right), \forall e \neq 0, \Delta$$

Tuning law: $\dot{\hat{l}}_1 = \gamma(\tau_1 + \tau_2)$

Control Law: $u = u_1 + u_2$

# Baseline controller for L1 adaptive controller

Coordinate transformation

$$x_1 = l_{os}$$
$$x_2 = -\sigma V \sin(\theta_{os})$$
$$u = \tan(\delta)$$

Simplified tractor error model with lumped disturbances

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (f(x) + G(x)u) + \begin{pmatrix} \Delta_1(x,w) \\ \Delta_2(x,w) \end{pmatrix} \quad (1)$$

- Assume $l_1$ is an unknown parameter with positive sign
- $f(x) = \frac{V^2 - x_2^2}{R_1 + x_1}$
- $G(x) = -\frac{\sigma V}{L_1}\sqrt{V^2 - x_2^2}$

Introduce baseline controller $u = k_x x + u_{ad}, k_x = \begin{pmatrix} \omega_n \\ 2\zeta\omega_n \end{pmatrix}$

$$\dot{x} = A_m x + b_m\left(\bar{f}(x) + G(x)u\right) + \Delta \quad (2)$$

- $\bar{f}(x) = f(x) + \left(1 + G(x)\right)k_x^T x$
- $A_m = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}$

# Linear Parametrization of Nonlinear systems and Linear Time-Varying representations of systems with unmodelled Actuator Dynamics

Approximate the nonlinear functions $\bar{f}(x)$ and $G(x)$ to fit the $\mathcal{L}_1$ architecture

$$\dot{x} = A_m x + b_m\left(\bar{f}(x) + G(x)u\right)$$

Lemma A.8.1. Let $x(t)$ be a continuous and piecewise differentiable function of $t \geq 0$. If $\|x_\tau\|_{\mathcal{L}\infty} \leq \rho$ and $\|\dot{x}_\tau\|_{\mathcal{L}\infty} \leq d_x$ for $\tau \geq 0$, where $\rho$ and $d_x$ are some positive constants, then there exists continuous $\theta(t)$ and $\sigma(t)$ with continuous derivative such that for all $t \in [0, \tau]$

$$\bar{f}(x) = \theta_t(t)\|x\|_\infty + \sigma_t(t) \qquad (3)$$

where

$$\left\|\frac{\partial f(t,x)}{\partial x}\right\|_1 \leq d_{f_x}(\delta), \qquad \left|\frac{\partial f(t,x)}{\partial t}\right| \leq d_{f_t}(\delta)$$

(conditions on Lemma A.8.1 hold for most of $(x_1, x_2)$-plane and around (0,0))

Partial knowledge of the system input gain

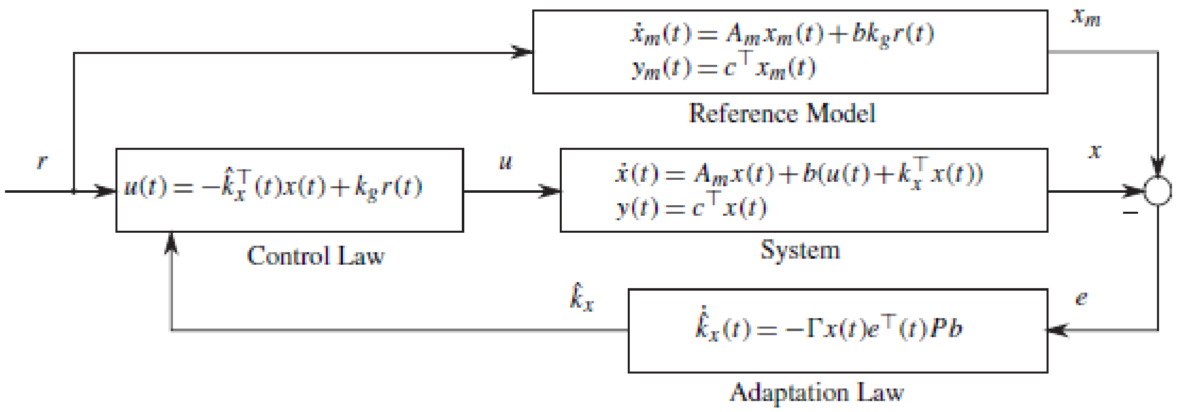
$$G(x) = \omega(t) \in [\omega_l, \omega_u] = \left[\epsilon, \frac{V^2}{L_1}\right], \qquad \sigma = -1 \qquad (4)$$

Architecture suited for controller

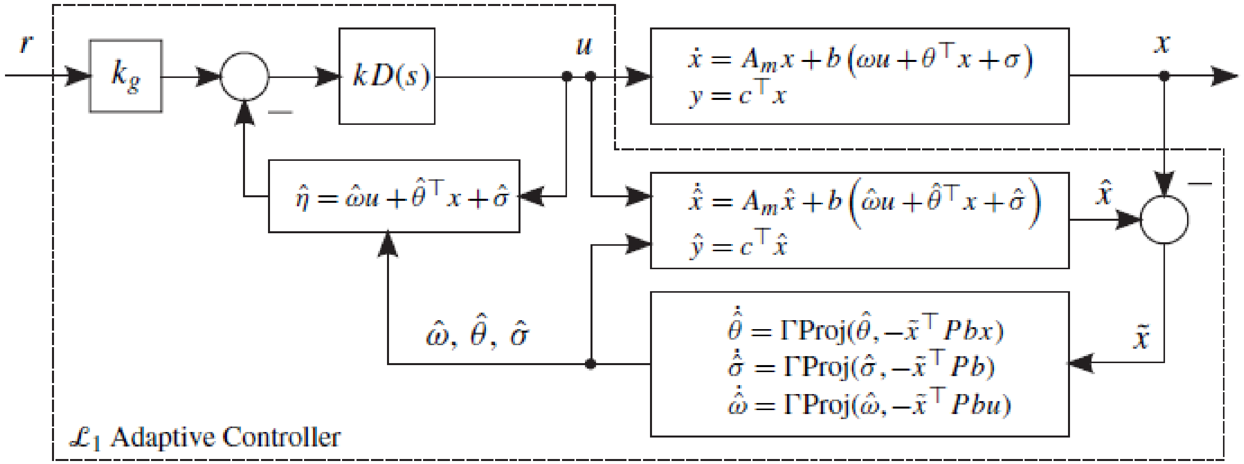
$$\dot{x} = A_m x + b_m(\omega(t)u + \theta_t(t)\|x\|_\infty + \sigma_t(t)\ ) \qquad (5)$$

# L1 adaptive controller architecture

## MRAC




## $\mathcal{L}_1$-adaptive control




$x_m(t)$ ⟷ $x(t)$ ⟷ $x_{ref}(t)$

$$\dot{x}_m = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} x_m + \begin{pmatrix} 0 \\ \omega_n^2 \end{pmatrix} r(t)$$

$$\dot{x}_{ref} = A_m x_{ref} + b\left(\bar{f}(x_{ref}) + g(x_{ref}) * u_{ref}\right)$$

$$u_{ref}(s) = \frac{C(s)}{\omega}\left(k_g r(s) - \eta_{ref}(s)\right)$$

- Tracking performance varies with $\Gamma$ adaptation gain
- High frequency adaptation leads to high frequency control signal
- $\tau_d$ time delay margin decreases with $\Gamma$
- Adaptive parameters unbounded given input disturbance

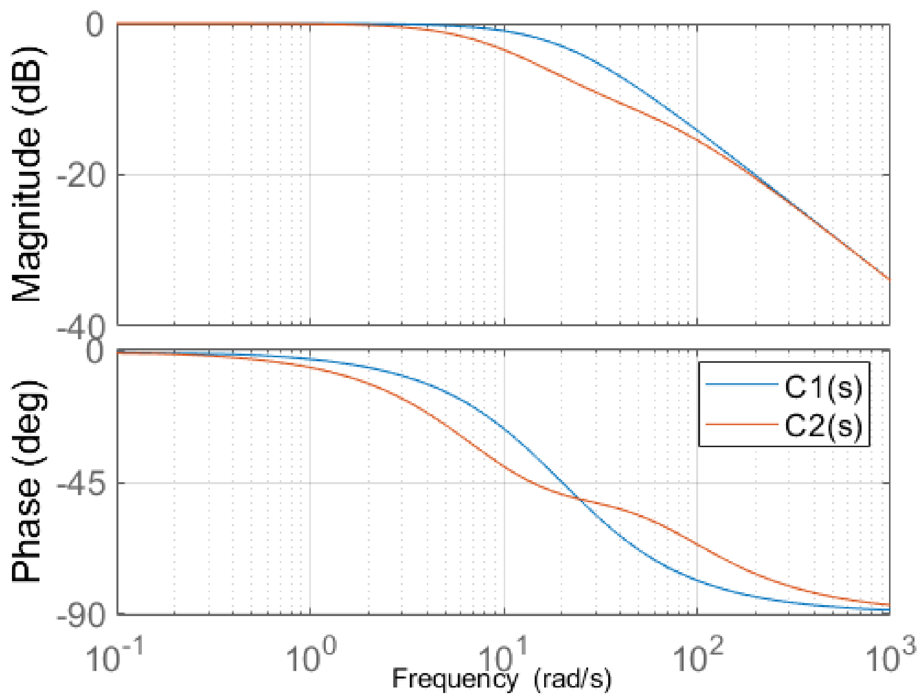
- Guaranteed transients $\|x - x_{ref}\|_{\mathcal{L}_1} \leq O\left(\frac{1}{\sqrt{\Gamma}}\right)$
- $\omega_c$ bandwidth of $C(s)$ limits high frequency components in $u(t)$
- $\tau_d$ time delay margin decoupled from $\Gamma$
- Adaptive parameters adjust to guarantee transients

# L1-norm condition and filter design $C(s) = \frac{\omega k D(s)}{1+\omega k D(s)}$, $k = 20$

$$H(s) = (s\mathbb{I} - A_m)^{-1} b_m$$
$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)}$$
$$G(s) = \mathrm{H}(s)\big(1 - C(s)\big)$$

$\mathcal{L}_1$-norm condition

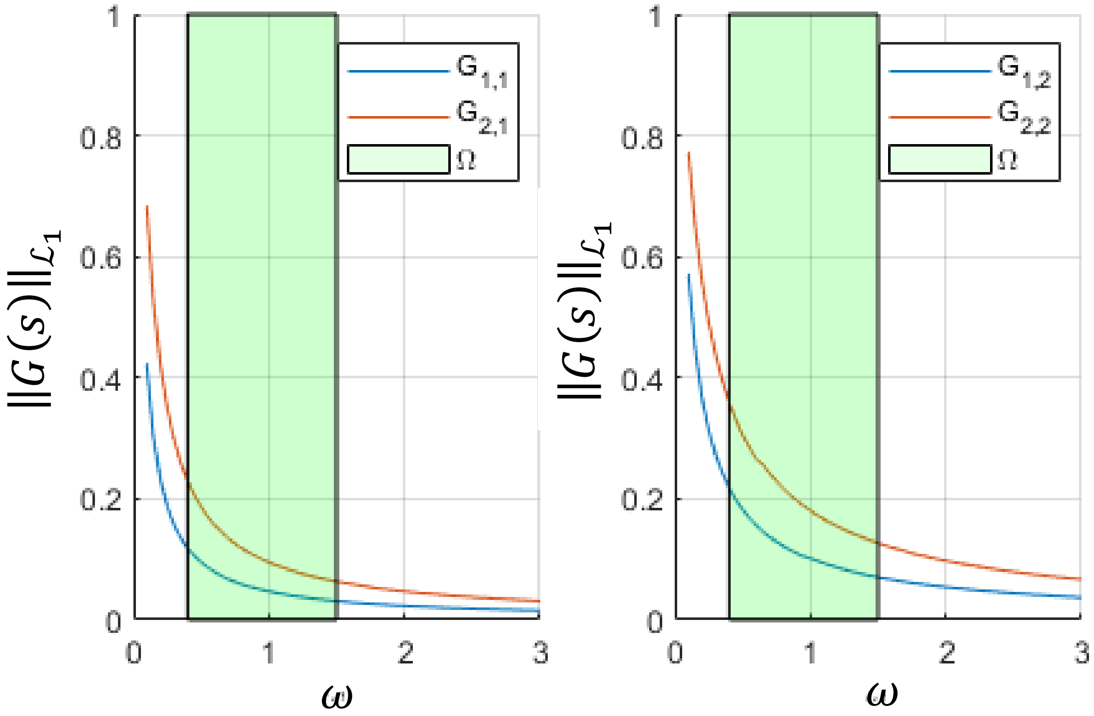
$$\|G(s)\|_{\mathcal{L}_1} L < 1$$



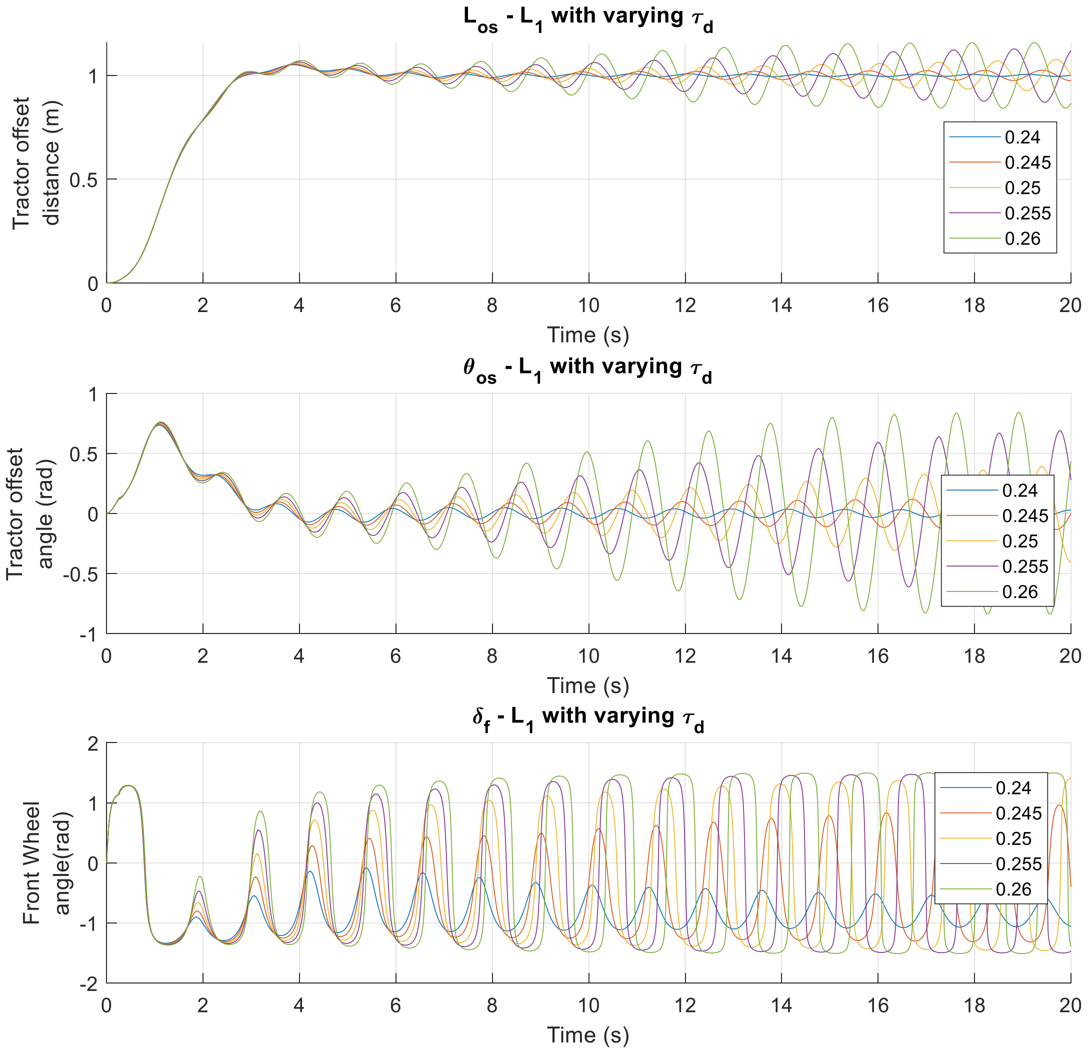

$$L = \max_{\Theta} \theta \,, \|G(s)\|_{\mathcal{L}_1} \equiv \|g_\tau\|_{\mathcal{L}_1} = \int_0^\tau |g(t)|dt \approx \sum_{k=1}^{n} g(k\Delta t)\Delta t$$

$$D_1(s) = \frac{1}{s} \rightarrow C_1(s) = \frac{\omega k}{s + \omega k}$$

$$D_2(s) = \frac{s + 30}{s(s + 60)} \rightarrow C_2(s) = \frac{\omega k s + 30\omega k}{s^2 + (60 + \omega k)s + 30\omega k}$$

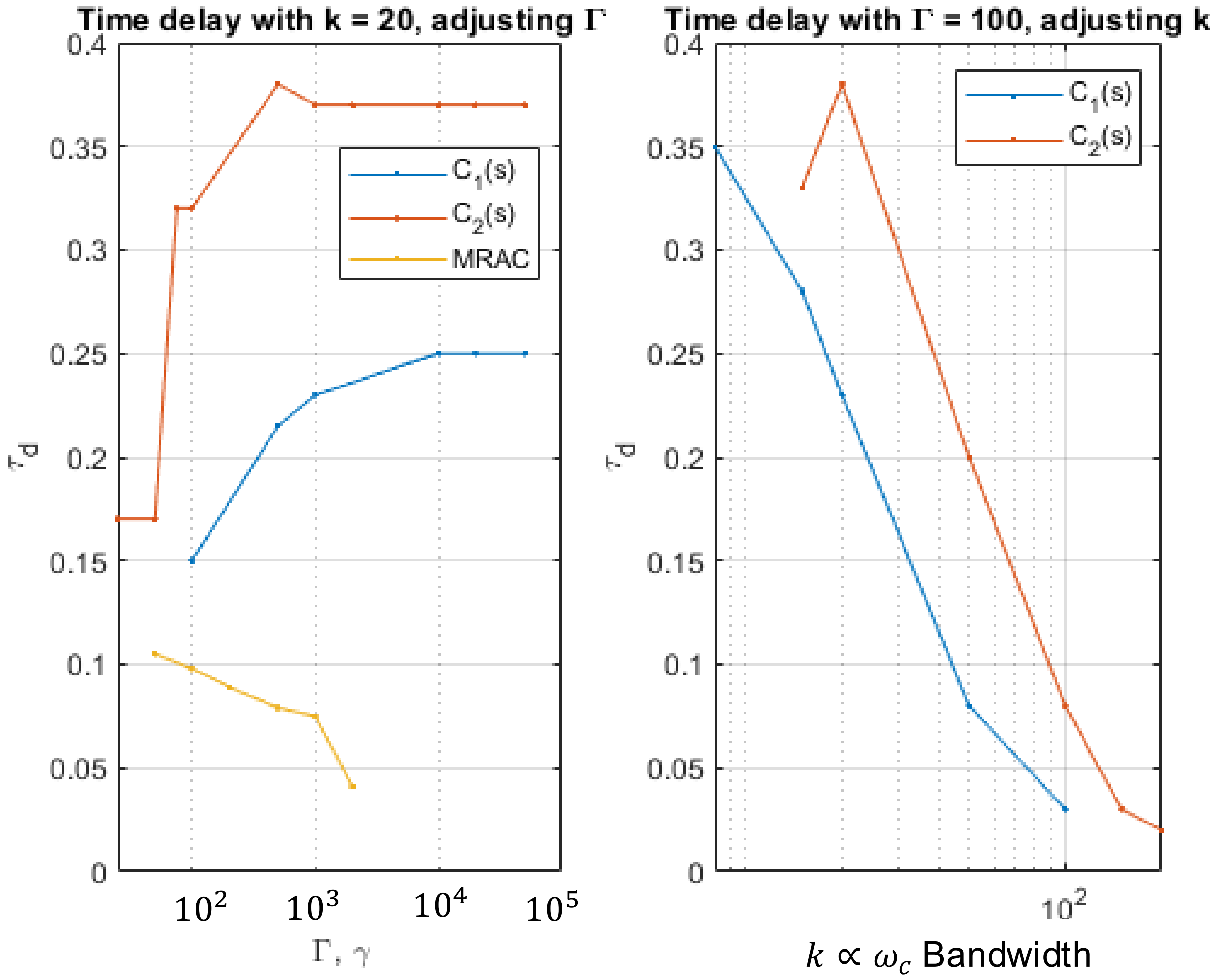


# Example search for time delay margin for $C_1(s), k = 50, \Gamma = 1000$




- Time delay margin based on when oscillations in $(l_{os}, \theta_{os})$ start to grow

# Time delay margin for varying $k$, $C_i(s)$, $\Gamma$ for MRAC and $\mathcal{L}_1$-adaptive control



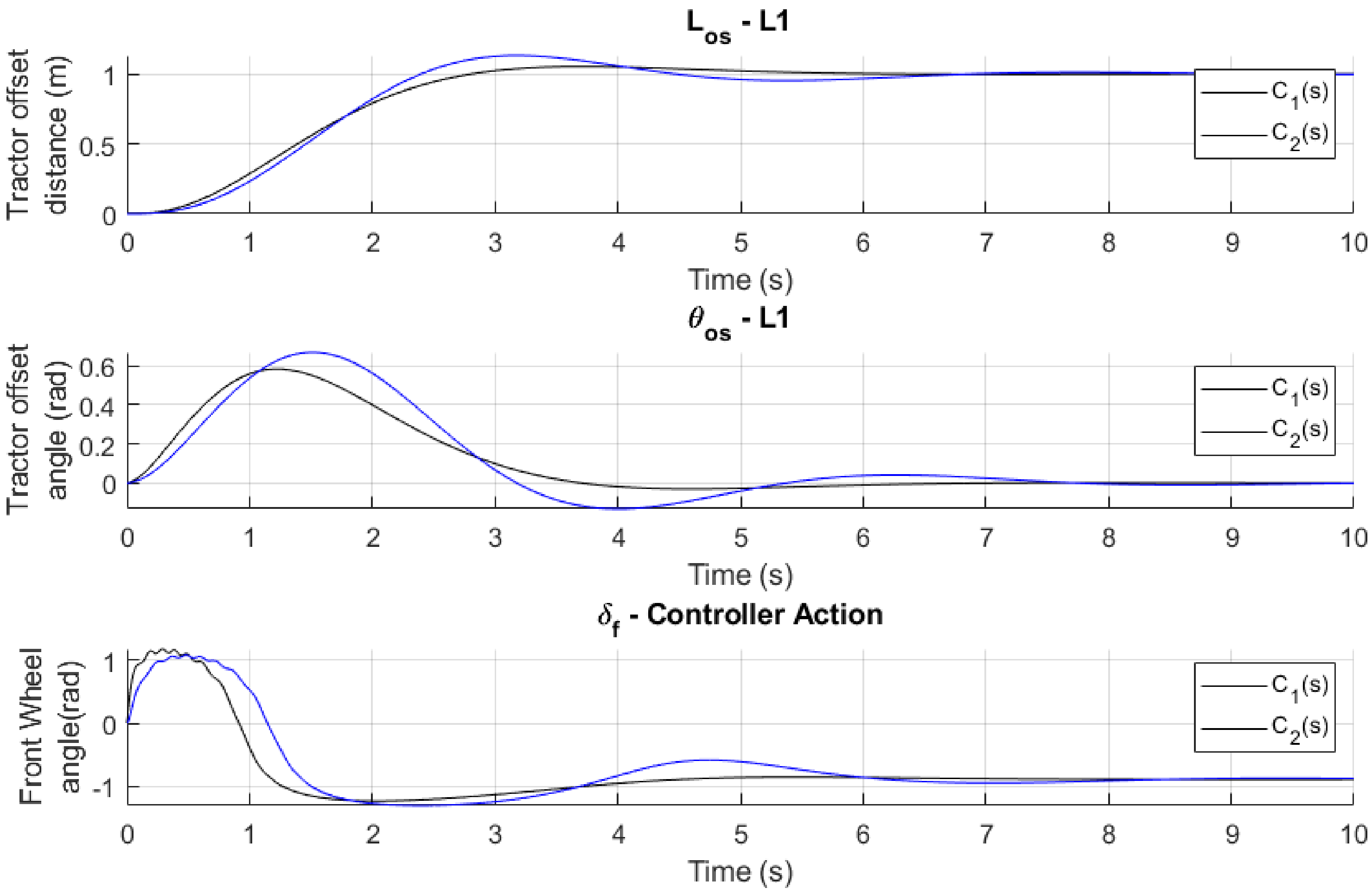

## $\mathcal{L}_1$ time delay margin becomes constant at high $\Gamma$

- Below $\Gamma \approx 25$, system goes unstable
- Much higher than MRAC time delay margin
- Higher-order filter $C_2(s)$ improves time delay margin at expense of performance

## High bandwidth (increasing $k$, $C(s) = \frac{kD(s)}{1+kD(s)}$) leads to loss of time delay margin

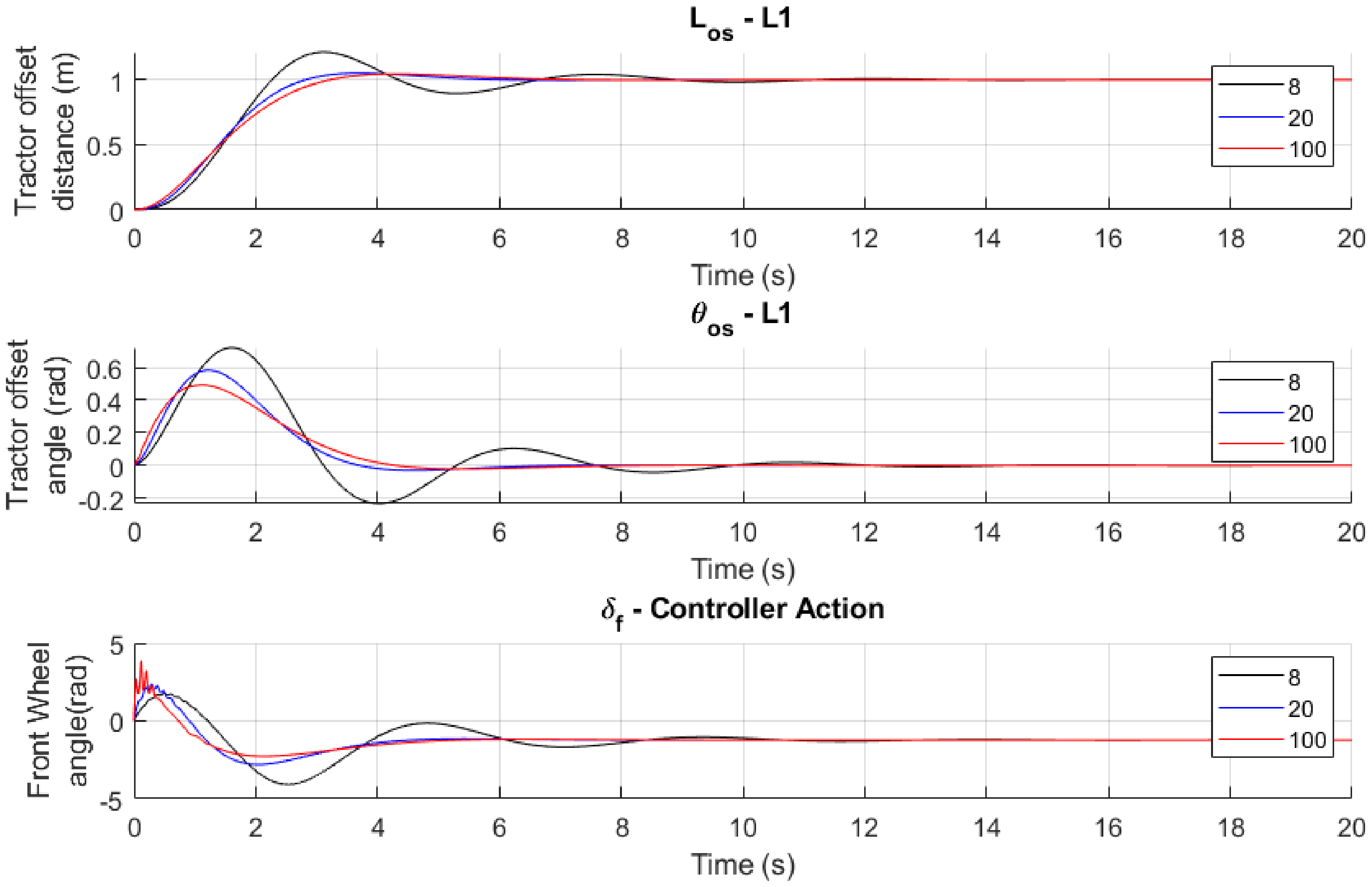
- Increasing bandwidth improves tracking performance

# Example comparing performance between $C_1(s)$ and $C_2(s)$, $k = 20$, $\Gamma = 1000$



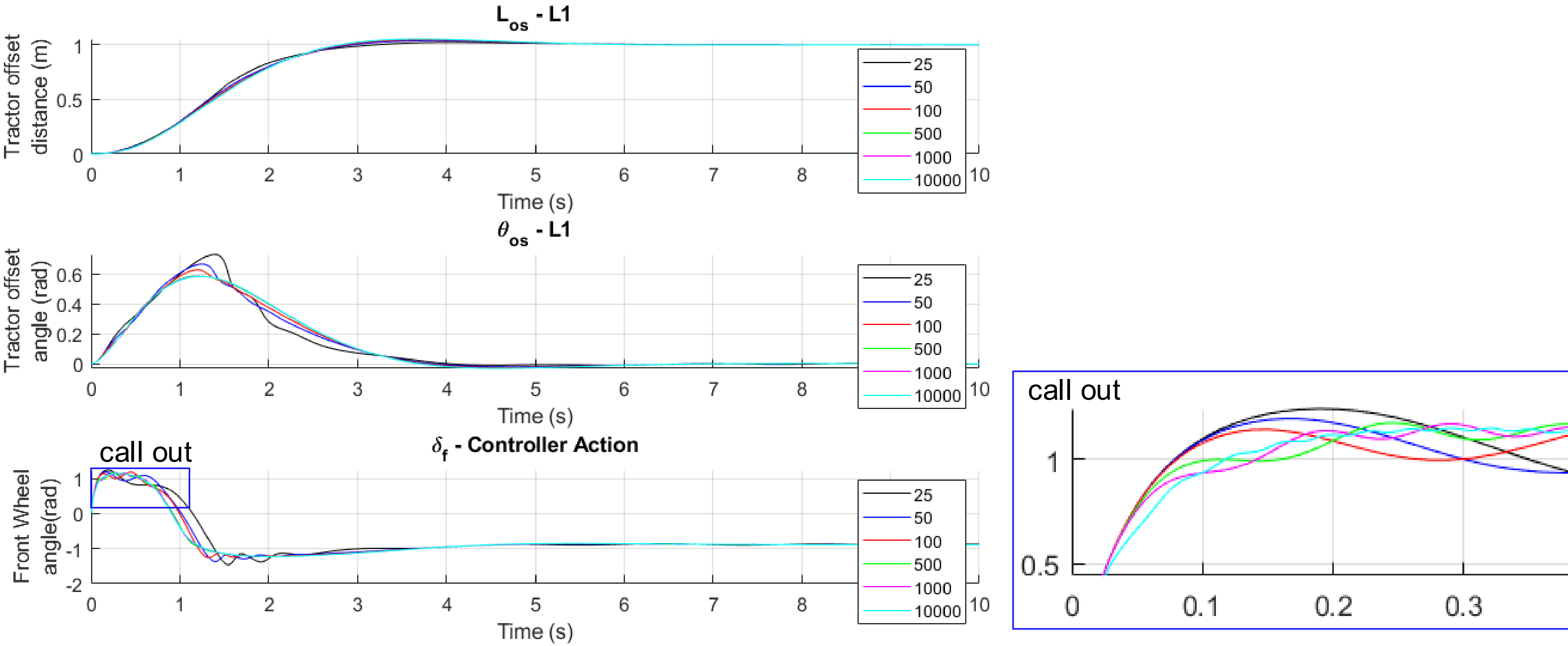

- Example of slower transient for $C_2(s)$ for same $k$, $\Gamma$
- Higher order filter trades some performance for robustness ($C_2(s)$ has higher $\tau_d$)

# Increasing bandwidth $k$ of filter improves tracking (transient speed)



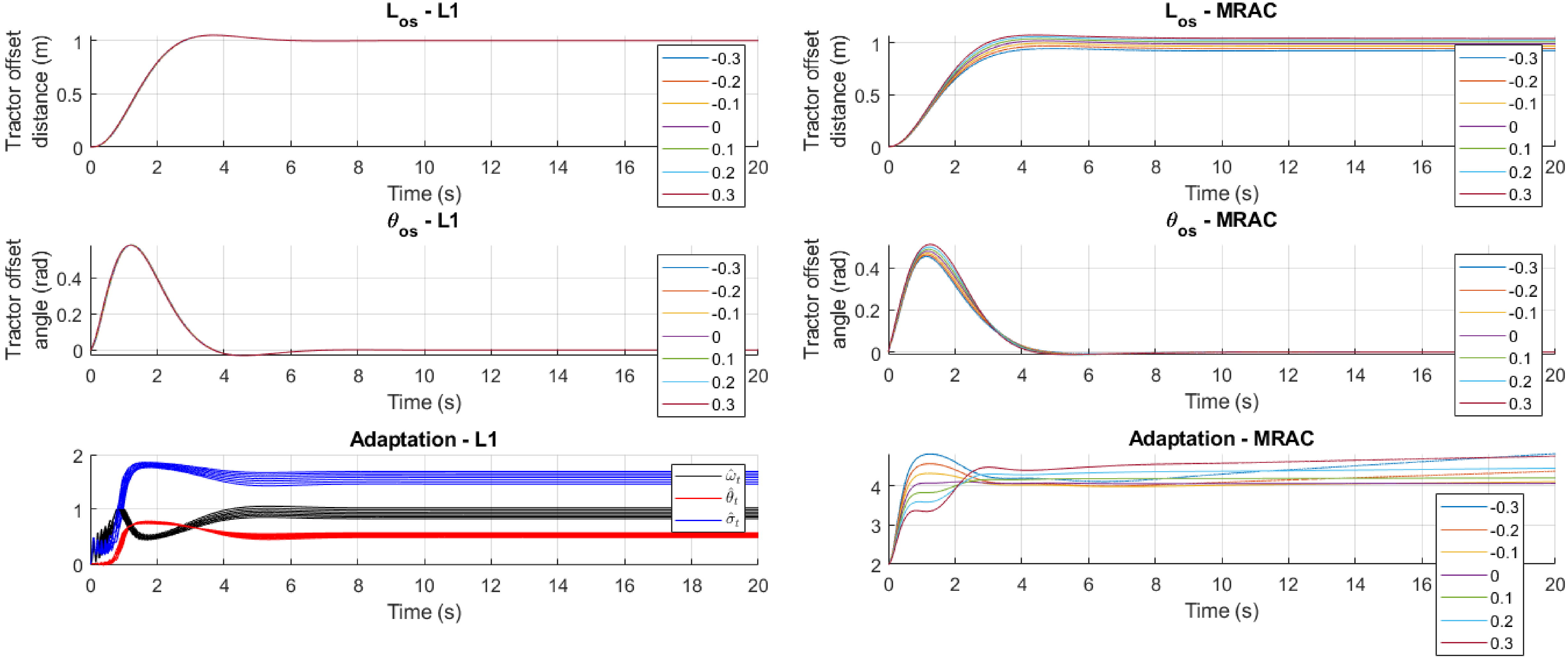

- At higher bandwidth, high frequency components noticeable in control signal
- Below $k \approx 8$, system is unstable as $\mathcal{L}_1$-norm condition fails

# Increasing $\Gamma$ adaptive gain improves tracking for $\mathcal{L}_1$



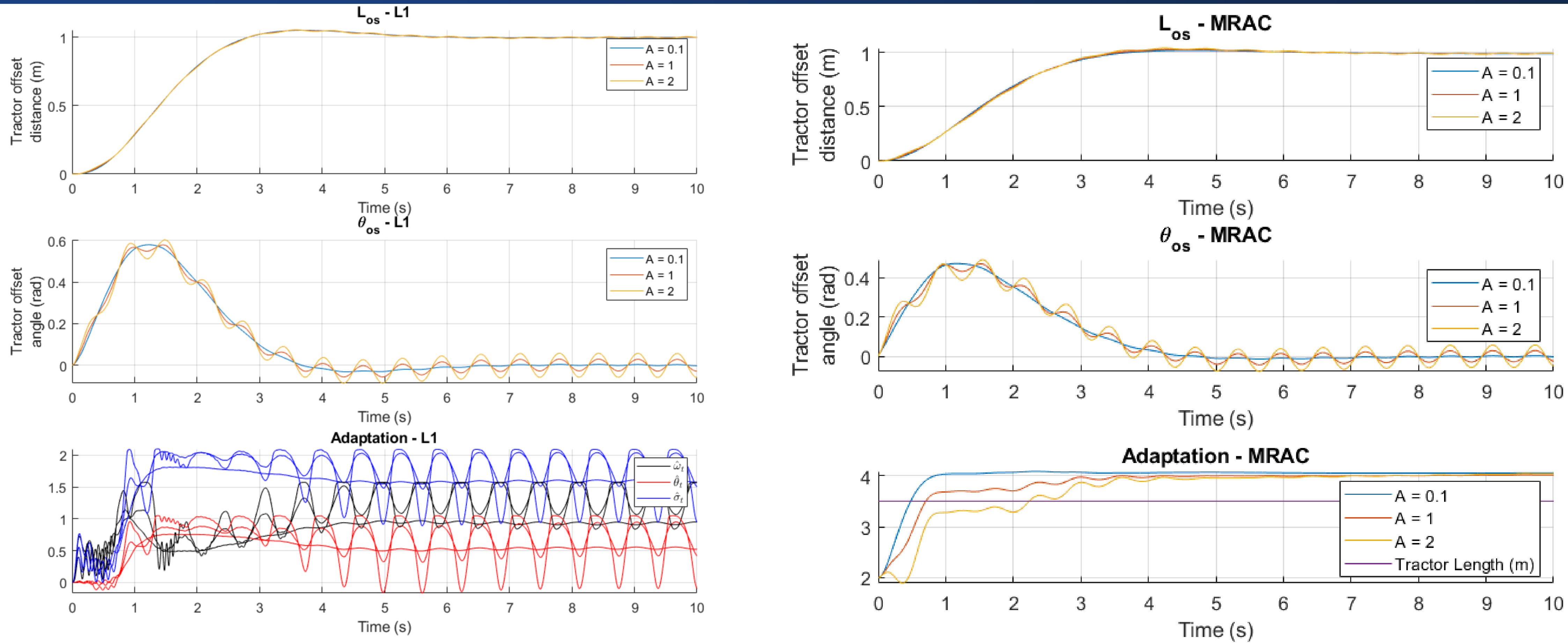

- ➢ Higher frequency components more noticeable in control signal with higher gain (still strongly attenuated) – control signal largely decoupled from $\Gamma$
- ➢ Insignificant increase in performance above $\Gamma = 500$

# System subject to constant input disturbance $u(t) = u_{ad}(t) + k_x^T x + C, \delta = \tan^{-1}(u(t))$



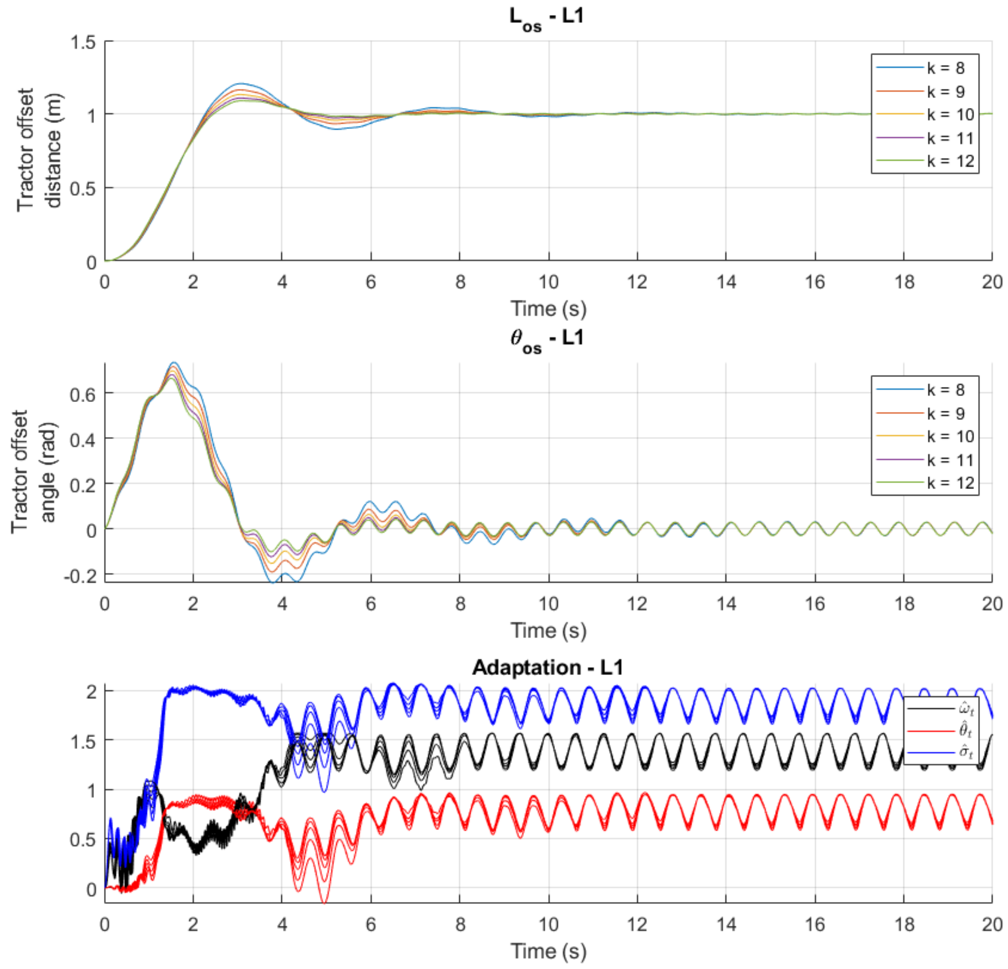

- ➢ $\mathcal{L}_1$ adaptive parameter steady state values adjust
- ➢ MRAC adaptive parameters slowly diverge, becomes extremely stiff

# Comparing transient performance between MRAC and $\mathcal{L}_1$-adaptive control for sinusoidal disturbance without retuning of $\Gamma$, $u(t) = u_{ad}(t) + k_x^T x + A\sin(10t)$



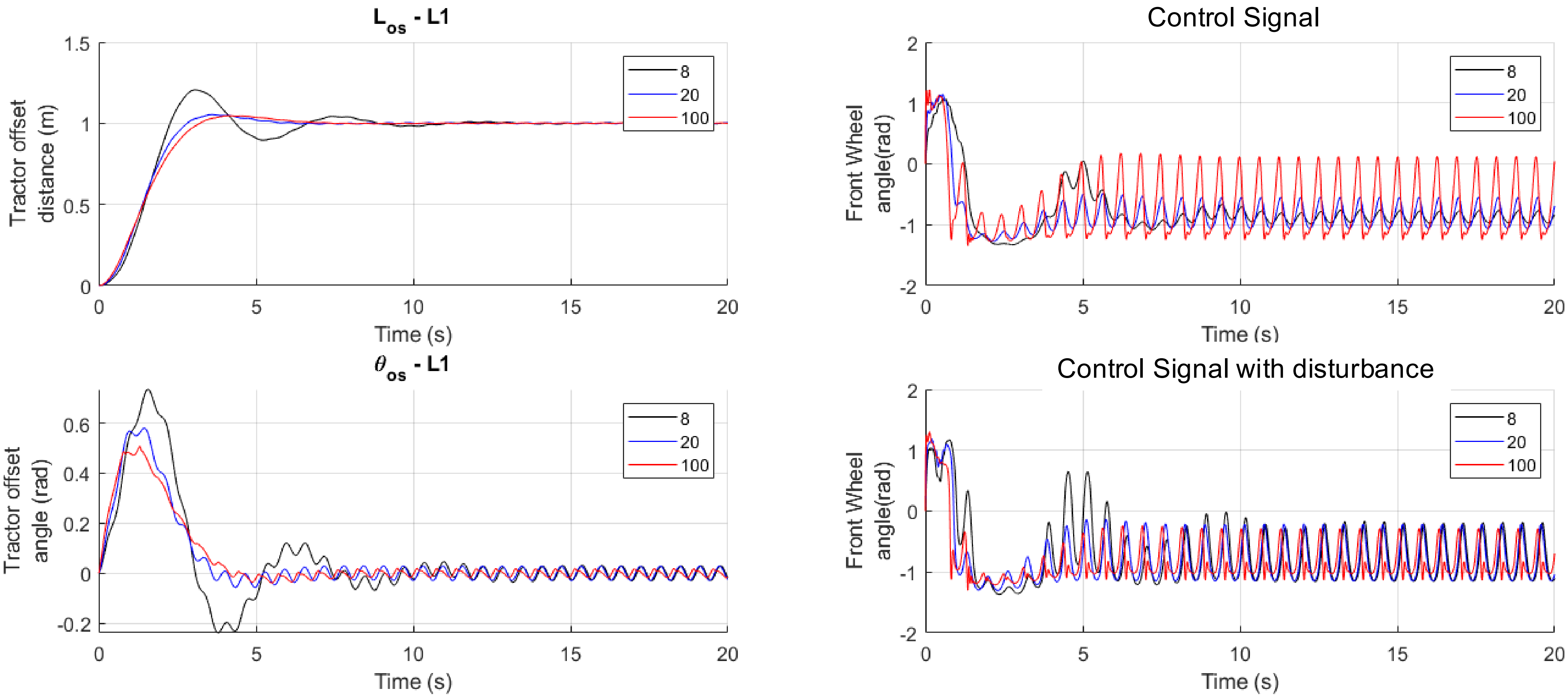

- $\mathcal{L}_1$-adaptive controller keeps adaptive parameters bounded due to projection – equations become excessively stiff for larger sinusoidal disturbances
- Indirect MRAC scheme designed with ISS-Backstepping has some robustness to disturbances (design parameter $\mu$) – Still has unbounded adaptive parameters

# Adjusting bandwidth $k$ subject, system subject to mid-frequency input disturbance $u(t) = u_{ad}(t) + k_x^T x + sin(10t), \delta = \tan^{-1}(u(t))$



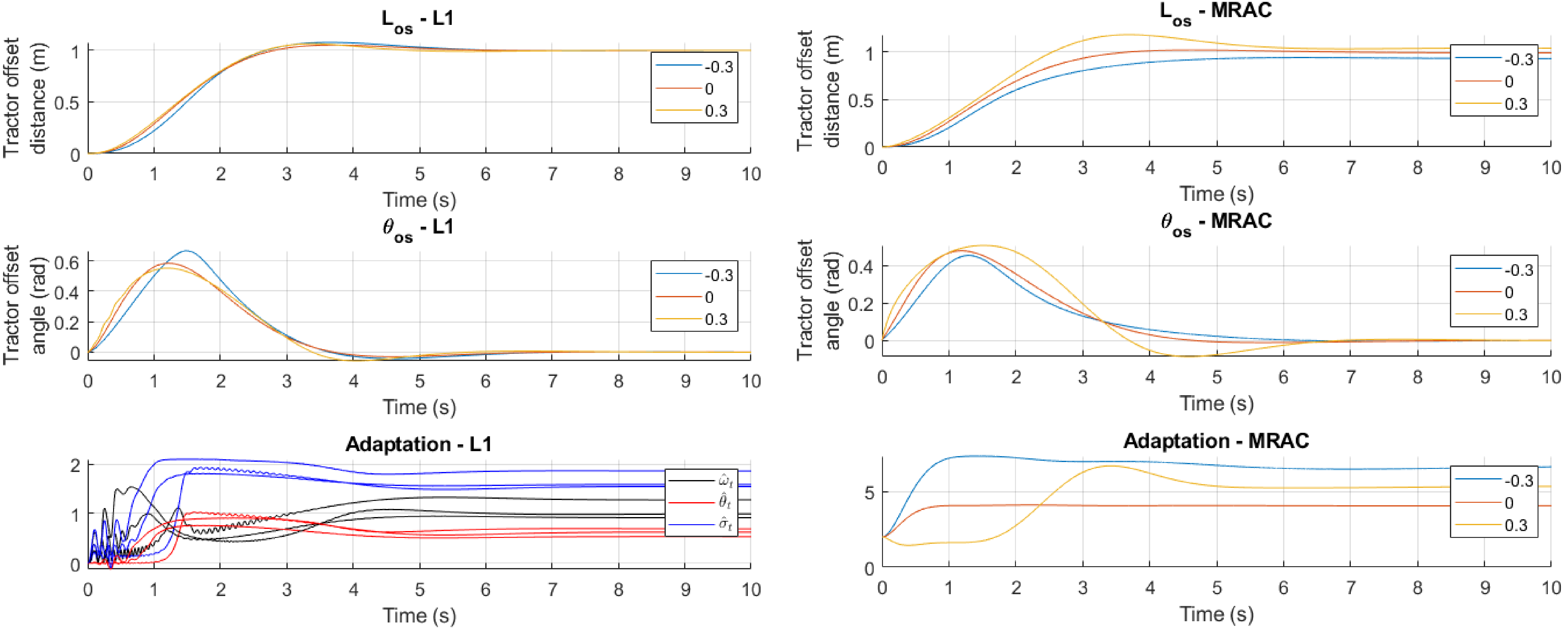

- Minimal improvement in tracking beyond $k = 10$ to disturbance $\sin(10t)$

# Effects of input disturbance $\sin(10t)$ and bandwidth on control signal $u_{ad}(t)$ and $u(t)$



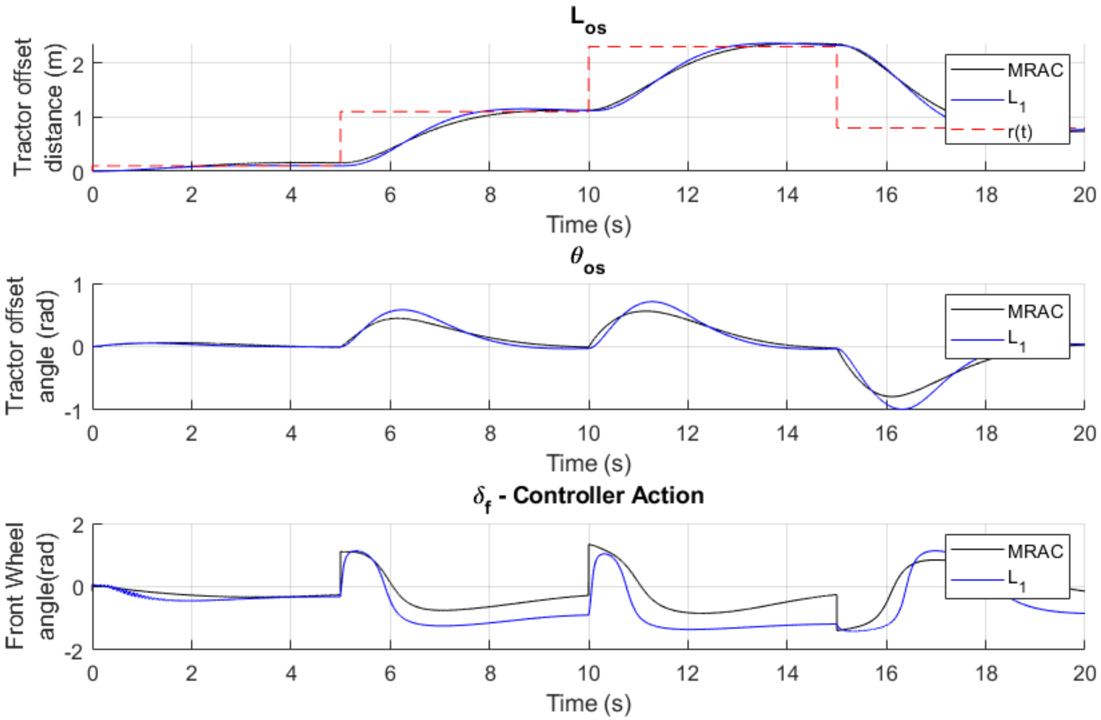

- ➢ For all bandwidths, control signal $u(t)$ similar although amplitude of $u_{ad}(t)$ differ

# Comparing response of MRAC and $\mathcal{L}_1$-adaptive controller to state disturbances $\Delta \beta_f \in [-0.3, 0, 0.3]$

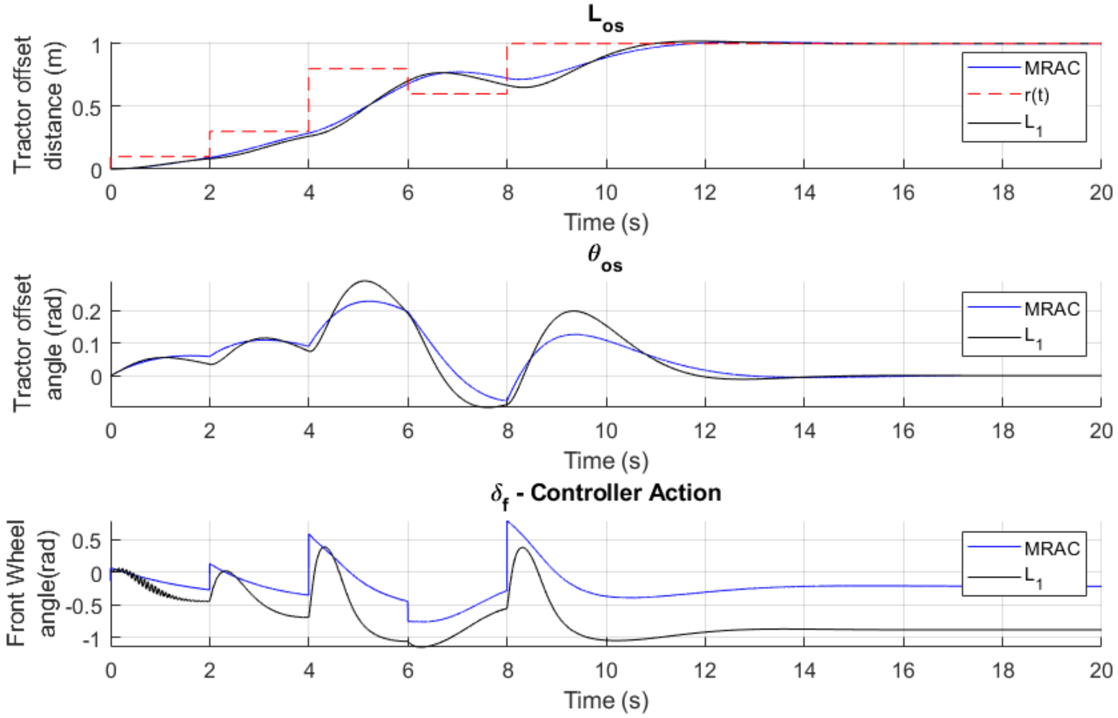



- $\mathcal{L}_1$adaptive controller can tolerate front wheel disturbances better than MRAC
- $\mathcal{L}_1$adaptive controller unstable beyond MRAC $|\beta_f| \approx 0.5$ rad, MRAC unstable beyond $|\beta_f| \approx 0.3$ rad

# Minimal difference in tracking a changing reference $r(t)$ for constant $\Gamma$ between MRAC controller and $\mathcal{L}_1$-adaptive controller




$r_1(t)$




$r_2(t)$

- Expect transient performance of MRAC controller to vary significantly for different reference inputs
- Equations become stiff for MRAC for more frequency piece-wise jumps in reference $r(t)$ while $\mathcal{L}_1$-adaptive controller has no such trouble
  - Step-change in control signal at step-change in reference $r(t)$

# Summary of observations and Conclusions

## $\mathcal{L}_1$-adaptive control

- Decouples $\Gamma$ from $u_{ad}(t)$ using LPF $C(s)$
  - $\Gamma$ (↑) has minimal effect on time delay margin $\tau_d$
  - Bandwidth $k$ (↑) improves tracking, allowing higher frequencies in $u_{ad}(t)$, reduces time delay margin $\tau_d$ (↓)
- Higher time delay margin $\tau_d$ for all cases tested than MRAC
- Lower frequency control signals for same performance, same input disturbances $u_d$ and same piece-wise continuous references $r(t)$
- Guaranteed transient envelope $\|x - x_{ref}\|$ which can be adjusted by tuning LPF $C(s)$ and $\Gamma$

## MRAC

- High adaptive gain $\Gamma$ leads to high frequency inputs $u(t)$
  - Robustness (↓) and performance (↑) with $\Gamma$ (↑)
- Minimal time delay margin $\tau_d$ which decreases with $\Gamma$
- unbounded parameter drift to input disturbances $u_d(t)$
  - ISS-backstepping design may provide some robustness to input disturbances
- No guaranteed transients
  - Cannot guarantee performance when $r(t)$ changes

- $\mathcal{L}_1$ adaptive control would help where transients risk destroying crops at corners of fields
- Possesses sufficient robustness to surface property and topological disturbances

# References

[1] Huynh, V.T., (2011), Model Development and Control for an Agricultural Tractor-trailer under the Influence of Slip. *Masters Thesis, School of Mechanical and Manufacturing Engineering, The University of New South Wales.*

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